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INTERLACING THEOREM FOR THE LAPLACIAN SPECTRUM OF A GRAPH

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Abstract. It is well known that the Interlacing theorem for the Laplacian spectrum of a finite graph and its induced subgraphs is not true in a general case. In this paper we completely describe all simple finite graphs for which this theorem is true. Besides, we prove a variant of the Interlacing theorem for Laplacian spectrum and induced subgraphs of a graph which is true in general case.

1. Introduction

First we repeat in short some elementary facts about the Laplacian spectrum of a finite graph which we shall use in the sequel.

Let G be a simple graph on n vertices and the vertex set $V(G) = \{v_1, \dots, v_n\}$. Next, let $A(G) = [a_{ij}]$ be its $(0, 1)$ adjacency matrix, and $D(G) = \operatorname{diag}(d_1, \dots, d_n)$ be the diagonal matrix with vertex degrees $d_1, \dots, d_n$ of its vertices $v_1, \dots, v_n$. Then $L(G) = D(G) - A(G)$ is called the Laplacian matrix of the graph G. It is symmetric, singular and positively definite. Its eigenvalues are all real and nonnegative and form the Laplacian spectrum $\sigma_L(G) = \{\lambda_1, \dots, \lambda_n\}$ of the graph G. We shall always assume that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. It is well known that $\lambda_n = 0$ and the multiplicity of 0 equals to the number of (connected) components of G. Hence, $\lambda_k(G) = 0$ for some $k = 1, \dots, n$ if and only if G has at least $n - k + 1$ components.

Theorem A. *If H is a (not necessary induced) subgraph of a finite graph G then*

$$\lambda_k(H) \leq \lambda_k(G) \qquad (k = 1, \dots, |H|).$$

Next, let $G_1 = (V(G_1), E(G_1)), \dots, G_m = (V(G_m), E(G_m))$ $(m \geq 2)$ be finite graphs with mutually disjoint sets of vertices $V(G_1), \dots, V(G_m)$. Then the direct sum $G = G_1 + \cdots + G_m$ of these graphs is defined by $V(G) = V(G_1) \cup \cdots \cup V(G_m)$ and $E(G) = E(G_1) \cup \cdots \cup E(G_m)$.

Theorem B. *If $G = G_1 + \cdots + G_m$ is the direct sum of graphs $G_1, \dots, G_m$, then*

$$\sigma_L(G_1 + \cdots + G_m) = \sigma_L(G_1) \cup \cdots \cup \sigma_L(G_m),$$

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including the multiplicities too.

Theorem C. *If $\overline{G}$ is the complementary graph of a graph G, then*

$$\lambda_k(\overline{G}) = n - \lambda_{n-k}(G) \qquad (k = 1, \ldots, n-1).$$

If G is a graph and H is any its induced subgraph, we shall denote it by $H \subseteq G$. The void graph on n vertices (without any edge) is denoted by E_n, the complete graph on n vertices is denoted by K_n, and the star on n vertices is denoted by $K_{1,n-1}$. The graph $K_2 + \cdots + K_2$ (p copies of the graph K_2) is denoted simply by pK_2.

2. Main results

By analogy to the known Interlacing theorem for the ordinary spectrum of a finite graph, we formulate a possible variant of the Interlacing theorem for the Laplacian spectrum of a graph. We shall call it "L.I.T." in short (the "Laplacian Interlacing Theorem").

L.I.T. *If G is a finite graph of order n ($n \in N$), then for every its induced subgraph H of order m ($m < n$), it holds*

$$\lambda_{n-m+k}(G) \le \lambda_k(H) \le \lambda_k(G) \qquad (k = 1, \ldots, m). \tag{1}$$

Note that by Theorem A the right-side of (1) is always true, even for an arbitrary subgraph H of G. Hence, the only interesting part of L.I.T. are in fact the inequalities

$$\lambda_k(H) \ge \lambda_{n-m+k}(G) \qquad (k = 1, \ldots, m). \tag{2}$$

Unfortunately, such a general theorem is, as is well known, not true in the general case. There are many counter–examples, and we notice only one.

Let $G = K_{1,n}$ ($n \ge 2$) be the star with n rays, and H be the induced subgraph $E_n \subseteq G$ obtained by removal the central vertex of G. Then

$$\sigma_L(G) = \{n+1, \underbrace{1, \ldots, 1}_{n-1}, 0\}, \quad \sigma_L(H) = \{\underbrace{0, \ldots, 0}_{n}\},$$

so that (2) obviously fails, because $\lambda_1(H) = 0 < \lambda_2(G) = 1$.

Therefore, we pose the following question:

Find all finite graphs G such that L.I.T. holds for G.

The next theorem completely resolves this question.

Theorem 1. *A graph G satisfies L.I.T. if and only if $G = G(p,q) = pK_2 + E_q$ for some integers $p, q \ge 0$ ($p + q \ge 1$).*

Proof. First suppose that G is an arbitrary graph of the form $G(p,q)$ ($p+q \ge 1$). The Laplacian spectrum of $G(p,q)$ reads:

$$\sigma_L(G(p,q)) = \{\underbrace{2, \ldots, 2}_{p}, \underbrace{0, 0, \ldots, 0}_{p+q}\}.$$

If H is any proper induced subgraph of G, then it is also of the form $G(p_0, q_0)$ ($p_0 + q_0 \ge 1$), where obviously $p_0 \le p$. Since the number of components of $G(p,q)$

is $p+q$, and a removal of any number of vertices of the graph $G(p,q)$ together with the corresponding edges does not increase the number of components, we conclude that $p+q \geq p_0+q_0$. Next, we have that

$$\sigma_L(H) = \{\underbrace{2,\dots,2}_{p_0},\underbrace{0,\dots,0}_{p_0+q_0}\},$$

so that obviously

$$\lambda_k(H) = 2 \geq \lambda_{n-m+k}(G) \quad (k=1,\dots,p_0).$$

Here $n = 2p+q$, $m = 2p_0+q_0 < n$.

Further, we have that $\lambda_k(H) = 0$ $(k = p_0+1,\dots,m)$, and $\lambda_{n-m+k}(G) = 0$ $(k = p_0+1,\dots,m)$ since $n-m+k > n-m+p_0 \geq p$, because $p+q \geq p_0+q_0$, as we have already said.

Hence, the inequalities (2) hold for every $k = 1,\dots,m$.

Conversely, let G satisfies L.I.T., and let $G_1,\dots,G_r$ $(r \geq 1)$ be the (connected) components of G. We first wish to prove that each component G_i is a complete graph $(i = 1,\dots,r)$.

On the contrary, suppose that for instance G_1 is not complete. Let v_1', v_1'' be two nonadjacent vertices in G_1, and $v_2 \in V(G_2),\dots,v_r \in V(G_r)$ be arbitrary fixed vertices. Then $v_1', v_1'', v_2,\dots,v_r$ form an induced subgraph $H \subseteq G$ which is void, so by (2) we find that

$$\lambda_1(H) = 0 \geq \lambda_{n-(r+1)+1}(G) = \lambda_{n-r}(G),$$

thus $\lambda_{n-r}(G) = 0$. But the last equality means that that G has at least $r+1$ components, what is a contradiction. Hence, all components $G_1,\dots,G_r$ are complete graphs. Without loss of generality we can assume that for some $p \geq 0$ $G_1 = K_{n_1},\dots,G_p = K_{n_p}$ $(n_1,\dots,n_p \geq 2)$ and $G_i = K_1$ $(i = p+1,\dots,r)$, so that $G = K_{n_1}+\cdots+K_{n_p}+E_q$ $(p+q = r \geq 1)$. We can also assume that $2 \leq n_1 \leq n_2 \leq \cdots \leq n_p$.

Next, we wish to prove that $n_1 = 2$. We obviously have that

$$\sigma_L(G) = \{\underbrace{n_p,\dots,n_p}_{n_p-1},\dots,\underbrace{n_1,\dots,n_1}_{n_1-1},\underbrace{0,\dots,0}_{p+q}\}.$$

Suppose on the contrary that $n_1 \geq 3$. Removing a vertex from the component K_{n_1}, we obtain an induced subgraph $H \subseteq G$, and

$$\sigma_L(H) = \{\underbrace{n_p,\dots,n_p}_{n_p-1},\dots,\underbrace{n_1-1,\dots,n_1-1}_{n_1-2},\underbrace{0,\dots,0}_{p+q}\}.$$

Since $n_1-2 \geq 1$, by (2) we easily get a contradiction $n_1-1 \geq n_1$.

Therefore $n_1 = 2$. Continuing this reasoning, we consecutively find that $n_2 = 2,\dots n_p = 2$, so that $G = G(p,q) = pK_2+E_q$ where $p+q \geq 1$. This completes the proof. $\square$

Finally, we formulate another variant of the Interlacing theorem which is more appropriate to the Laplacian spectrum of a graph, and is true in the general case.

Theorem 2. *If G is a graph of order n and H is any its induced subgraph of order m $(m < n)$, then it holds:*

$$\lambda_{n-m+k}(G) - n + m \le \lambda_k(H) \le \lambda_k(G) \qquad (k = 1, \ldots, m). \tag{3}$$

Proof. We only need to prove the left inequalities in (3).

First, it is obviously true for $k = m$ because $\lambda_n(G) = \lambda_m(H) = 0$ and $n > m$.

Next, assume that $k \le m - 1$. Denoting by $\overline{G}$ the complement of G and by $\overline{H}$ the complement of H, we have that $\overline{H}$ is an induced subgraph of $\overline{G}$, so that

$$\lambda_k(\overline{H}) \le \lambda_k(\overline{G}) \qquad (k = 1, \ldots, m-1).$$

But since $\lambda_k(\overline{G}) = n - \lambda_{n-k}(G)$ and $\lambda_k(\overline{H}) = m - \lambda_{m-k}(G)$ $(k = 1, \ldots, m-1)$, we find that

$$m - \lambda_{m-k}(H) \le n - \lambda_{n-k}(G) \qquad (k = 1, \ldots, m-1).$$

Replacing k with $m - k$, we get

$$\lambda_k(H) \ge \lambda_{n-m+k}(G) - n + m \qquad (k = 1, \ldots, m-1),$$

and finally

$$\lambda_k(H) \ge \lambda_{n-m+k}(G) - n + m \qquad (k = 1, \ldots, m).$$

□

Obviously, the above inequalities have a sense only for values $k \le m$ such that $\lambda_{n-m+k}(G) \ge n - m$.

Also notice that the previous proof can not be used if H is an arbitrary subgraph of a graph G, since in this case $\overline{H}$ is not necessary a subgraph of the graph $\overline{G}$. Moreover, this statement is again not true for subgraphs of a graph in the general case.

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